# Problem 4.46

Consider the **three-dimensional harmonic oscillator**, for which the potential is

$$V(r) = \frac{1}{2}m\omega^2 r^2. \tag{4.215}$$

**(a)** Show that separation of variables in cartesian coordinates turns this into three one-dimensional oscillators, and exploit your knowledge of the latter to determine the allowed energies. *Answer*:

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega. \tag{4.216}$$

**(b)** Determine the degeneracy $d(n)$ of $E_n$.

---

## Solution

Schrödinger's equation governs the time evolution of a wave function.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

Here the aim is to solve it in all of space with the potential energy function,

$$V(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2).$$

Since $V = V(x,y,z)$, expand the Laplacian operator in Cartesian coordinates.

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right) + V(x,y,z)\Psi(x,y,z,t), \quad -\infty < x,y,z < \infty,\ t > 0$$

Because Schrödinger's equation and its associated boundary conditions ($\Psi \to 0$ as $|\mathbf{x}| \to \infty$) are linear and homogeneous, the method of separation of variables can be applied: Assume a product solution of the form $\Psi(x,y,z,t) = X(x)Y(y)Z(z)T(t)$ and plug it into the PDE.

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}[X(x)Y(y)Z(z)T(t)] = -\frac{\hbar^2}{2m}\bigg\{&\frac{\partial^2}{\partial x^2}[X(x)Y(y)Z(z)T(t)] \\ &+ \frac{\partial^2}{\partial y^2}[X(x)Y(y)Z(z)T(t)] \\ &+ \frac{\partial^2}{\partial z^2}[X(x)Y(y)Z(z)T(t)]\bigg\} \\ &+ \frac{1}{2}m\omega^2(x^2+y^2+z^2)[X(x)Y(y)Z(z)T(t)] \end{aligned}$$

$$\begin{aligned} i\hbar X(x)Y(y)Z(z)T'(t) = -\frac{\hbar^2}{2m}\left[X''(x)Y(y)Z(z)T(t) + X(x)Y''(y)Z(z)T(t) + X(x)Y(y)Z''(z)T(t)\right] \\ + \frac{1}{2}m\omega^2(x^2+y^2+z^2)[X(x)Y(y)Z(z)T(t)] \end{aligned}$$

Divide both sides by $X(x)Y(y)Z(z)T(t)$ in order to separate variables.

$$i\hbar\frac{T'(t)}{T(t)} = -\frac{\hbar^2}{2m}\left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right] + \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

The only way a function of $t$ can be equal to a function of $x$, $y$, and $z$ is if both are equal to a constant.

$$i\hbar\frac{T'(t)}{T(t)} = -\frac{\hbar^2}{2m}\left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right] + \frac{1}{2}m\omega^2(x^2+y^2+z^2) = E$$

Bring the terms with $y$ and $z$ to the right side.

$$-\frac{\hbar^2}{2m}\frac{X''(x)}{X(x)} + \frac{1}{2}m\omega^2x^2 = E + \frac{\hbar^2}{2m}\left[\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right] - \frac{1}{2}m\omega^2(y^2+z^2)$$

The only way a function of $x$ can be equal to a function of $y$ and $z$ is if both are equal to another constant.

$$-\frac{\hbar^2}{2m}\frac{X''(x)}{X(x)} + \frac{1}{2}m\omega^2x^2 = E + \frac{\hbar^2}{2m}\left[\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)}\right] - \frac{1}{2}m\omega^2(y^2+z^2) = F$$

Bring the terms with $y$ to one side.

$$-\frac{\hbar^2}{2m}\frac{Y''(y)}{Y(y)} + \frac{1}{2}m\omega^2y^2 = E - F + \frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)} - \frac{1}{2}m\omega^2z^2$$

The only way a function of $y$ can be equal to a function of $z$ is if both are equal to another constant.

$$-\frac{\hbar^2}{2m}\frac{Y''(y)}{Y(y)} + \frac{1}{2}m\omega^2y^2 = E - F + \frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)} - \frac{1}{2}m\omega^2z^2 = G$$

As a result of using the method of separation of variables, Schrödinger's equation has reduced to four ODEs—one in $x$, one in $y$, one in $z$, and one in $t$.

$$\left.\begin{aligned} i\hbar\frac{T'(t)}{T(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{X''(x)}{X(x)} + \frac{1}{2}m\omega^2x^2 &= F \\ -\frac{\hbar^2}{2m}\frac{Y''(y)}{Y(y)} + \frac{1}{2}m\omega^2y^2 &= G \\ E - F + \frac{\hbar^2}{2m}\frac{Z''(z)}{Z(z)} - \frac{1}{2}m\omega^2z^2 &= G \end{aligned}\right\}$$

The strategy is to solve the second and third eigenvalue problems for $F$ and $G$, then to solve the fourth eigenvalue problem for $E$, and then finally to solve the first eigenvalue problem for $T(t)$.

Since $-\infty < x, y, z < \infty$, the method of operator factorization can be used to solve the latter three (as was done in Problem 2.10).

$$F = \left(j + \frac{1}{2}\right)\hbar\omega, \qquad j = 0, 1, 2, \ldots$$

$$G = \left(k + \frac{1}{2}\right)\hbar\omega, \qquad k = 0, 1, 2, \ldots$$

$$E - F - G = \left(l + \frac{1}{2}\right)\hbar\omega, \qquad l = 0, 1, 2, \ldots$$

Substitute the formulas for $F$ and $G$ into the third equation.

$$E - \left(j + \frac{1}{2}\right)\hbar\omega - \left(k + \frac{1}{2}\right)\hbar\omega = \left(l + \frac{1}{2}\right)\hbar\omega$$

Solve for $E$.

$$E_{jkl} = \left(j + k + l + \frac{3}{2}\right)\hbar\omega, \quad \begin{cases} j = 0, 1, 2, \ldots \\ k = 0, 1, 2, \ldots \\ l = 0, 1, 2, \ldots \end{cases}$$

Evaluate the energy for many values of $j$, $k$, and $l$.

$$E_{000} = \left(0 + 0 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{3}{2}\right)\hbar\omega = E_0$$

$$E_{100} = \left(1 + 0 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{5}{2}\right)\hbar\omega = E_1$$

$$E_{010} = \left(0 + 1 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{5}{2}\right)\hbar\omega$$

$$E_{001} = \left(0 + 0 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{5}{2}\right)\hbar\omega$$

$$E_{110} = \left(1 + 1 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega = E_2$$

$$E_{101} = \left(1 + 0 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega$$

$$E_{011} = \left(0 + 1 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega$$

$$E_{200} = \left(2 + 0 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega$$

$$E_{020} = \left(0 + 2 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega$$

$$E_{002} = \left(0 + 0 + 2 + \frac{3}{2}\right)\hbar\omega = \left(\frac{7}{2}\right)\hbar\omega$$

Evaluate the energy for more values of $j$, $k$, and $l$.

$$E_{111} = \left(1 + 1 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega = E_3$$

$$E_{210} = \left(2 + 1 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{201} = \left(2 + 0 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{120} = \left(1 + 2 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{021} = \left(0 + 2 + 1 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{012} = \left(0 + 1 + 2 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{102} = \left(1 + 0 + 2 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{300} = \left(3 + 0 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{030} = \left(0 + 3 + 0 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

$$E_{003} = \left(0 + 0 + 3 + \frac{3}{2}\right)\hbar\omega = \left(\frac{9}{2}\right)\hbar\omega$$

Therefore, following the pattern,

$$\boxed{E_n = \left(n + \frac{3}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots.}$$

The degeneracy is the number of states that have the same energy. As a result, $d_0 = 1$, $d_1 = 3$, $d_2 = 6$, and $d_3 = 10$. Notice that to get $d_1$, 2 needs to be added to $d_0$; to get $d_2$, 3 needs to be added to $d_1$; and to get $d_3$, 4 needs to be added to $d_2$. The pattern is apparent for $d_{n+1}$.

$$d_{n+1} = (n + 2) + d_n, \quad d_0 = 1$$

This is a recurrence relation, more specifically an inhomogeneous first-order linear difference equation with constant coefficients. Bring $d_n$ to the left side.

$$d_{n+1} - d_n = n + 2$$

The left side is how the discrete derivative of a function $d_n$ of the integers is defined.

$$Dd_n = n + 2$$

Take the discrete antiderivative of both sides by summing from 0 to $n-1$.

$$\sum_{q=0}^{n-1} Dd_q = \sum_{q=0}^{n-1}(q+2)$$

$$Dd_0 + Dd_1 + Dd_2 + \cdots + Dd_{n-2} + Dd_{n-1} = \sum_{q=0}^{n-1} q + \sum_{q=0}^{n-1} 2$$

$$(d_1 - d_0) + (d_2 - d_1) + \cdots + (d_{n-1} - d_{n-2}) + (d_n - d_{n-1}) = 0 + \sum_{q=1}^{n-1} q + 2\sum_{q=0}^{n-1} 1$$

$$d_n - d_0 = \frac{(n-1)[(n-1)+1]}{2} + 2[(n-1)+1]$$

$$d_n - 1 = \frac{(n-1)n}{2} + 2n$$

$$d_n - 1 = \frac{n^2+3n}{2}$$

$$d_n = \frac{n^2+3n+2}{2}$$

Therefore, the degeneracy of energy $E_n$ is

$$\boxed{d_n = \frac{(n+2)(n+1)}{2}.}$$